Princeton Competitive Programming

Dynamic Programming I

Pedro Paredes

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Outline

Introduction

What is DP?

Algorithm design technique based on breaking problems into simpler subproblems

Usual workflow:

- Break the problem into overlapping subproblems
- Solve each subproblem and store the answer
- Combine the subproblems into a solution to the main problem

Finding the right subproblem break down is part art part science

Can only be learned by looking at lots of examples

1D DP

Fibonacci

The well-known Fibonacci series is defined as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}$$

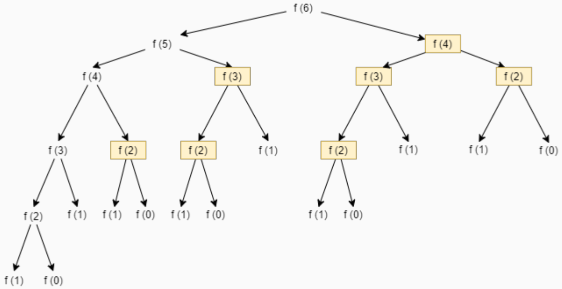
This is often used as an example of recursion like so:

```
public static int fib(int n) {
  if (n <= 1)
    return n;
  return fib(n - 1) + fib(n - 2);
}
```

Fibonacci

How efficient is this solution?

This is the recursive tree:



Source: `https://textbooks.cs.ksu.edu/cc310/6-recursion/6-example-fibonacci-numbers/`

Tip: notice how the time complexity of this solution is related to the number of leaves, which in turn is related to the actual value of F_n

Fibonacci

To avoid repeating calculations we memoize (i.e. store) the values of each F_n after computing it:

```
public static int[] dp = new int[N];

public static int fib(int n) {
    if (n <= 1)
        return 1;

    if (dp[n] > 0)
        return dp[n];

    return dp[n] = fib(n - 1) + fib(n - 2);
}
```

This is clearly much faster, but by how much?

Fibonacci

Suppose we use the previous code to compute F_n, what is the time complexity in terms of n?

What happens when we call `fib(i)`:

- if we computed `fib(i)` before we return it, which is $O(1)$
- if not, we compute it and store it, which is $O(1)$

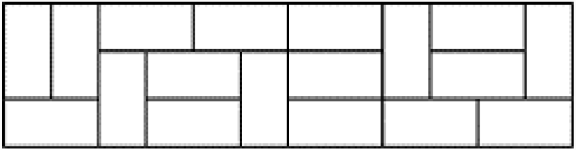
Since we compute each input once, the total time is $O(n)$

Tiling

In the Fibonacci example the problem subdivision was obvious: it was just the definition itself. So let's look at a more interesting one

Problem

Given an integer n, find the number of ways to fill a $3 \times n$ board with 1×2 dominoes

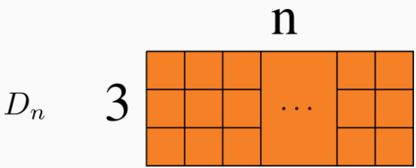


Example source http://web.stanford.edu/class/cs97si/04-dynamic-programming.pdf

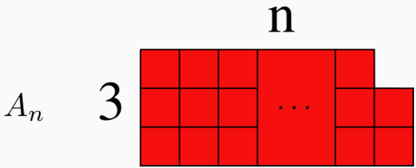
Tiling

We need to find a recurrence that breaks problem into subproblems

Let D_n be the number of ways of tiling a $3 \times n$ grid

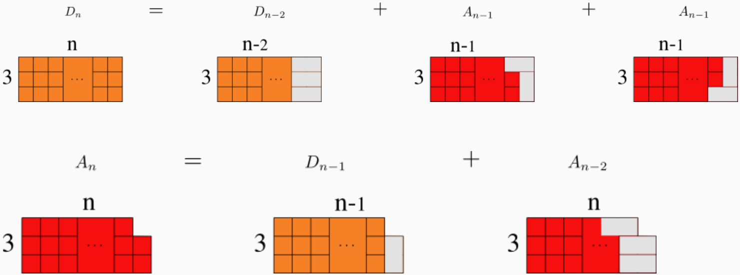


Let A_n be the number of ways of tiling a $3 \times n$ grid with a "hole" on the top-right corner



Tiling

Let's try to use the previous subproblems and define a recurrence based on trying to till the last column



Tiling

Based on the previous here is the recurrence:

```
public static int D(int n) {
    if (n == 0)
        return 1;
    if (n == 1)
        return 0;
    return D(n - 2) + 2 * A(n - 1);
}

public static int A(int n) {
    if (n <= 1)
        return 1;
    return D(n - 1) + A(n - 2);
}
```

Exercise: memoize the above code to make it $O(n)$

2D DP

Paths on a Grid

Problem

Given an n by n integer matrix, find a path from the upper-left corner to the lower-right corner with the lowest sum. The path can only move down or right.

3	7	9	2	7
9	8	3	5	5
1	7	9	8	5
3	8	6	4	10
6	3	9	7	8

The recurrence is the following:

$$dp(x, y) = \min(dp(x-1, y), dp(x, y-1)) + grid_{x,y}$$

Paths on a Grid

Based on the previous here is the solution:

```
int[][] dp = new int[n][n];

for (int y = 1; y <= n; y++) {
    for (int x = 1; x <= n; x++) {
        dp[y][x] = Math.min(dp[y][x - 1], dp[y - 1][x]) + grid[y][x];
    }
}
```

This is $O(n^2)$ since we have two nested for loops

Interval DP

Palindromic Edit Distance

Problem

Given a sequence of n characters $x_1 x_2 \ldots x_n$, find the minimum number of characters we need to add to make it a palindrome

If $x =$ `abga` we can add one `b` and make it `abgba`, so the answer is 1

We need to think about how to define some recurrence that is easy to compute

Palindromic Edit Distance

Let D_{ij} be the minimum number of characters that need to be inserted to make $x_i \dots x_j$ into a palindrome

So the solution is given by D_{1n}

$$D_{ij} = \begin{cases} 1 + \min\{D_{i+1,j}, D_{i,j-1}\} & x_i \neq x_j \\ D_{i+1,j-1} & x_i = x_j \end{cases}$$

Exercise: implement that to make it $O(n^2)$

Bitmask DP

Traveling Salesman Problem

Problem

Given a complete graph with n vertices, the cost between each pair of vertices u, v is a positive integer $c_{u,v}$. Find the minimum sum cycle that visits each vertex once.

Note that since we are looking for a cycle with all vertices, without loss of generality we can look for cycles that start at vertex 0 and end at vertex 0

Recursion idea

Suppose we have some partially built path that is currently at a vertex v and has visited all the vertices in a set S

Let `tsp(x, S)` be cheapest way to complete this path, i.e. cheapest path that starts at x, visits all vertices in $V \setminus S$ and ends at vertex 0

Traveling Salesman Problem

Note that we can define the following recursion to compute `tsp(x, S)`:

$$\text{tsp}(x, S) = \min_{y \notin S}\{\text{tsp}(y, S \cup \{y\}) + c_{xy}\}$$

And we have a base case when S contains all vertices, i.e. $S = V$:

$$\text{tsp}(x, V) = c_{x0}$$

Since here we have visited everyone so we have to return to 0

Traveling Salesman Problem - Detour: Bitmasks

How do we memoise over sets?

Note that n-bit integers represent sets: $\texttt{100100} \equiv \{0, 3\}$, $\texttt{000001} \equiv \{5\}$

So we can use 32-bit integers to represent sets of up to 32 elements

Some bitwise operations on sets (suppose b is a bitmask representing a set S and i is a vertex index):

- $\texttt{1} << \texttt{i}$ is the set $\{i\}$
- $(\texttt{1} << \texttt{N}) - 1$ is the set $\{0, \ldots, N-1\}$ (all numbers up to N)
- $\texttt{b}\&(\texttt{1} << \texttt{i})$ is 0 if $i \notin S$ and positive otherwise
- $\texttt{b}|(\texttt{1} << \texttt{i})$ adds i to S, so it is $S \cup \{i\}$

Traveling Salesman Problem

```
public static int[][] dp = new int[N][1 << N];

public static int tsp(int i, int S) {
    if (S == ((1 << N) - 1)) {
        return c[i][0];
    }
    if (dp[i][S] != -1) {
        return dp[i][S];
    }
    int res = Integer.MAX_VALUE;
    for (int j = 0; j < N; j++) {
        if ((S & (1 << j)) > 0)
            continue;
        res = Math.min(res, c[i][j] + tsp(j, S | (1 << j)));
    }
    return dp[i][S] = res;
}
```

The solution to the overall problem is `tsp(0, 1<<0)`

This is $O(2^n \cdot n^2)$