PEDRO PAREDES

Square Root Techniques

Princeton Competitive Programming

Basic Premise

- Separate something of size O(**n**) into blocks of size O(sqrt(**n**))
- Precompute something per block and then combine answers

Problem 1: Range Queries

Problem Description:

- You are given an array **A** with **n** integers and **Q** queries
- Each query asks for the maximum of the elements in interval $[\mathbf{l}_i, \mathbf{r}_i]$
- (Note: this works with any associative operation, max, min, sum, gcd, etc)

Problem Solution:

we'll pick **x** later

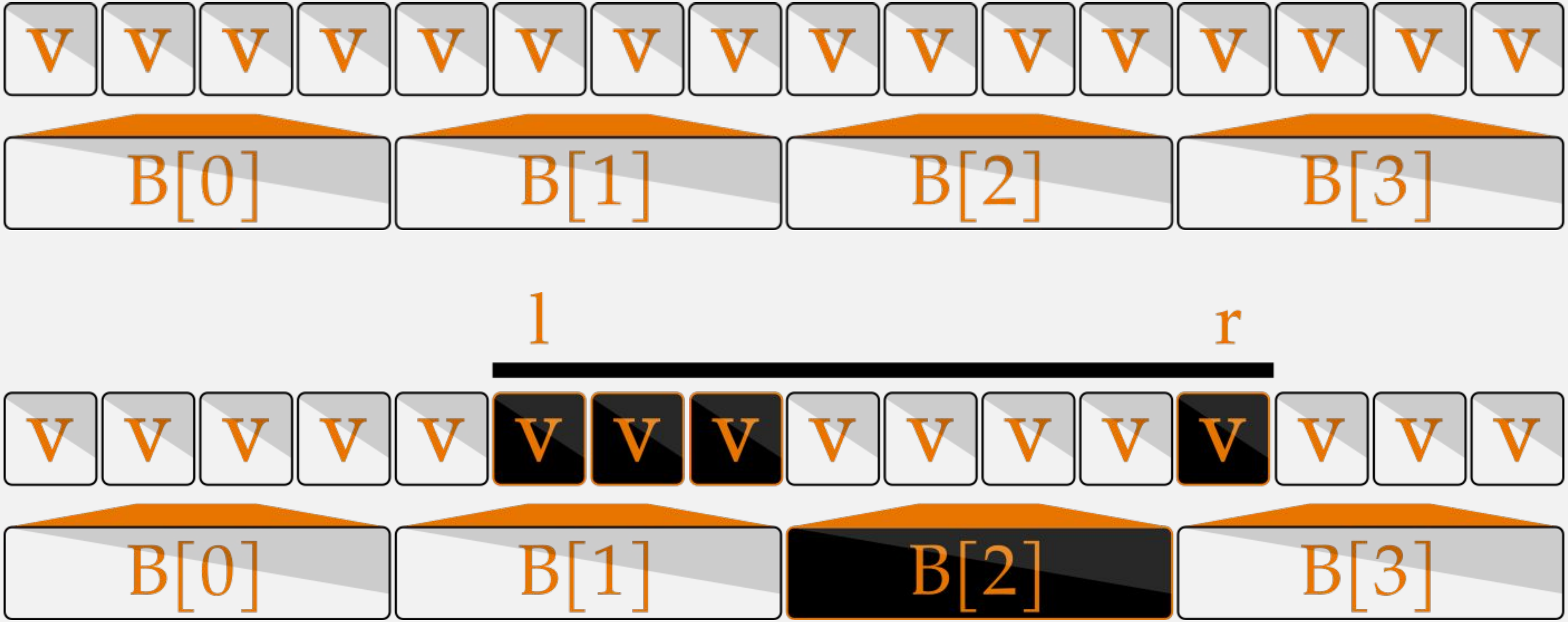
- Split **A** into **x** blocks of size **n/x** each (the last block might be smaller if **n** isn't divisible by **x**)
- Compute maximum of values in block **i** and store in an array **B**

$$\mathbf{B}[\mathbf{i}] = \max\{\mathbf{A}[\mathbf{xi}], \mathbf{A}[\mathbf{xi}+1], \ldots, \mathbf{A}[\mathbf{x}(\mathbf{i}+1)-1]\}$$

- Given an interval [**l**, **r**], decompose into blocks: there will be at most 2 partial blocks and **n/x** full blocks
- Compute maximum in partial blocks (costs O(**x**)) and use precomputed value for full blocks (costs O(**n/x**))
- Total cost per query is O(**x** + **n/x**) so pick **x**=sqrt(**n**) and we get a cost per query of O(sqrt(**n**))

Problem 1: Range Queries

Image of solution:



Problem 2: Range Queries with Updates

Problem Description:

- You are given an array **A** with **n** integers and **Q** queries **of two types**
- First query type asks for the maximum of the elements in interval [$\mathbf{l_i}$, $\mathbf{r_i}$]
- **Second query type asks to update the value of A[i] to v**

Problem Solution:

- Same block decomposition as before
- To update value of **A**[**i**] recalculate maximum of block containing value **i**
- The update cost is O(sqrt(**n**))

Problem 3: Range Queries with Range Updates

Problem Description:

- You are given an array **A** with **n** integers and **Q** queries of two types
- First query type asks for the **sum** of the elements in interval [l_i, r_i]
- Second query type asks to add **v** to each A[l_i … r_i]

Problem Solution:

- Same solution as before, but need two extra changes:
- To update partial blocks, update each **A**[**i**] individually and recalculate the value of **B**
- To update full blocks, add **block_size** * **v** to the corresponding block **B**[**i**]
- You also need to store a "lazy" value per block, so you can update the **A**[**i**] if you need to access any one individual value

Problem 4: Range Counting

Problem Description:

- You are given an array **A** with **n** integers and **Q** queries of two types
- First query type asks for the number of the elements in interval [l_i, r_i] that are equal to **y**
- Second query type asks to update the value of **A**[**i**] to **v**

Problem Solution:

- Divide into blocks as before
- Store a map/dictionary/hash table per block, storing the frequency of each element in the block
- To answer the first query, go through the partial blocks element by element and for full blocks query the map to determine the frequency of **v**
- Updating is the same, just update the map and the individual **A**[**i**]
- This takes time either O(**n***sqrt(**n**)*log(**n**)) or O(**n***sqrt(**n**)) with a hash map

Problem 5: Tree Updates

Problem Description:

- You are given a tree with **n** vertices each with a value $\mathbf{v}_i$, and **Q** queries of two types
- First query type asks for the value of vertex **i**
- Second query type asks to add **y** to all neighbors of vertex **i**

Problem Solution:

- Partition each node into one of two categories: heavy, if degree is > sqrt(**n**); light, if degree is < sqrt(**n**)
- Note that there are at most 2*sqrt(**n**) heavy nodes (since number of edges is < **n**)
- To do a query of the second type, if the node is light just go through all neighbors and add one by one. If the node is heavy, store a "lazy" extra value
- To do a query of the first type, add the extra values of all the heavy neighbors of **i** to its own value
- This takes time O(**Q***sqrt(**n**))

Problem 6: Grid Painting

Problem Description:

- Consider a square grid with **n** cells that are originally unpainted
- Process **n** queries each of which is a cell **c** of the grid
- For each query, first compute the distance to the closest painted cell and then paint that cell
- (Note: you can extend this problem to a tree instead of a grid, but it’s more technical)

Problem Solution:

- Given a certain state of the grid, with one BFS running in O(**n**) time we can determine the distance from each unpainted cell to the closest painted cell
- So now we can batch the queries into blocks of sqrt(**n**)
- At the start of each batch, use the BFS algorithm to compute distances
- For each query, go through each cell in the batch that came before (so at most sqrt(**n**) of them), calculate the distance to **c** and update if it is lower than the precomputed one
- In total we run sqrt(**n**) BFSs, and for each query we go through a list of length at most sqrt(**n**), so runtime is O(**n** * sqrt(**n**))

Problem 7: Balanced BST

Problem Description:

- Consider an array of integers that is initially empty and process **n** operations:
    - Add **x** to the array
    - Remove **x** from the array
    - Find if **x** is in the array

Problem Solution:

- We can solve this with a balanced BST, but suppose you don't know how to implement a balanced BST, but you know how to implement a normal BST
- Use an unbalanced BST and insert into it naively
- After sqrt(**n**) operations, reconstruct the BST so that it is now balanced
- Each reconstruction takes O(**n**) time, but we only do sqrt(**n**) of them, so this takes O(sqrt(**n**)) amortized time per operation

Problem 8: Offline Dynamic Connectivity

Problem Description:

- You are given an undirected graph **G** and you should process operations of three types:
    - Add an edge to the graph
    - Remove an edge from the graph
    - Check if nodes **u** and **v** are connected

Problem Solution:

- Group queries into batches of sqrt(**n**)
- First, fix all the edges that are contained in all sqrt(**n**) graphs of one batch of queries
- Collapse the graph into connected components using the fixed edges (using a DFS)
- Now for each query in one batch, add/remove the edge to the collapsed graph and run one DFS per check
- Since the collapsed graph has at most O(sqrt(**n**)) edges, each check runs in O(sqrt(**n**)) time

Further reading

Links:

https://assets.hkoi.org/training2023/sqrt.pdf

http://acm.math.spbu.ru/~sk1/mm/lections/mipt2016-sqrt/mipt-2016-burunduk1-sqrt.en.pdf

https://assets.hkoi.org/training2019/sqrt.pdf

https://codeforces.com/blog/entry/23005

https://codeforces.com/blog/entry/83248

https://usaco.guide/CPH.pdf#page=263